R. Vianna - Feb 3, 2011 - Lagrangian Floer homology

(M^{2n}, ω) symplectic manifold

$L \subset M$ Lagrangian: $\omega_{|L} = 0$, $\dim L = n$

<u>Ex:</u> $M = T^*X$, $\omega = d\alpha$,

φ 1-form on X $\rightsquigarrow$ graph(φ) = section of T^*X satisfies $\varphi^*\alpha = \varphi$
$\varphi : X \to T^*X$

graph(φ) is Lagrangian $\iff$ $\varphi^*\omega = \varphi^* d\alpha = d\varphi = 0$ $\iff$ φ is closed.

- Weinstein's nbd theorem: $L \subset M^{2n}$ Lagrangian
  $\Rightarrow$ $\exists$ nbd of L in M which is symplectomorphic to a nbd of the zero section in T^*L

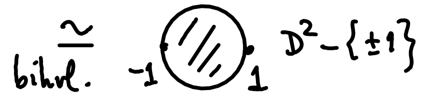
<u>Lagrangian Floer homology</u> $\quad L_0 \pitchfork L_1$ transverse Lagrangians (compact)
$\to L_0 \cap L_1$ = finite set

Novikov field $\Lambda = \{ \sum_i a_i T^{\lambda_i} \,/\, \lambda_i \in \mathbb{R},\ \lambda_i \to +\infty,\ a_i \in \mathbb{C} \}$
or some other field

Floer complex: $CF(L_0, L_1) = \Lambda^{|L_0 \cap L_1|}$

Fix J an almost-complex structure on M, $J \in End(TM)$, $J^2 = -1$, compatible with ω ie. $g_J = \omega(-, J\cdot)$ riemannian metric.

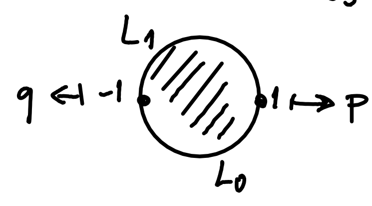
Differential will count J-holom. maps



st. • $\bar\partial_J u = 0$ ie. $\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0$

• finite energy $E(u) = \int u^*\omega = \iint \left| \frac{\partial u}{\partial s} \right|^2_{g_J} < \infty$

• at boundary, u maps



• Then define
$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u] \in \pi_2(M, L_0, L_1) \\ \text{st. } \operatorname{ind}([u]) = 1}} \#\left(\mathcal{M}(p,q,[u],J)/\mathbb{R}\right) T^{\omega(u)} q$$

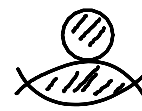
$\left[\omega(u) = \int u^*\omega = E(u)\right]$

Maslov index = trivializing u^*TM, and going from p to q, counts how many times TL_0 and TL_1 become non-transverse

$\longleftrightarrow$ expected $\dim_{\mathbb{R}}$ of $\mathcal{M}(p,q,[u],J) = \operatorname{ind}([u])$

<u>Issues:</u> - transversality of $\bar\partial$-operator ($\Rightarrow \mathcal{M}(p,q,[u],J)$ manifold)
Holds for generic choice of J

- compactness : need to compactify $\mathcal{M}$



$\left.\begin{array}{l}\to \text{sphere bubbling} \\ \to \text{disc bubbling}\end{array}\right\}$ assume don't happen
top. since disc bubbling prevents $\partial^2 = 0$

Then $\overline{\mathcal{M}}$ compactification by adding broken strips

- orientation

<u>Thm:</u> (Floer) If $[\omega].\pi_2(M) = 0$ ($\Rightarrow$ no J-holomorphic spheres / discs) and $[\omega].\pi_2(M, L_i) = 0$
then for generic J, ∂ is well-defined, $\partial^2 = 0$, and $HF(L_0, L_1)$ is independent of J, and invariant under Hamiltonian isotopy $HF(L_0, L_1) = HF(L_0, \psi(L_1))$

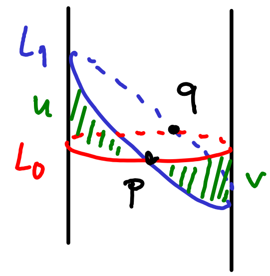
• Ham. isotopy inv^{ce} $\Rightarrow$ can define $HF(L,L) := HF(L, \psi(L))$ for $\psi \in \text{Ham}$.

<u>Thm:</u> $HF^*(L,L) \simeq H^*(L;\Lambda)$

So: $\psi(L) \pitchfork L \Rightarrow |\psi(L) \cap L| \geq \sum b_i(L)$ (Arnold's conj.)

(still assuming that L doesn't bound J-hol discs).

Example: $T^*S^1 = S^1 \times \mathbb{R}$



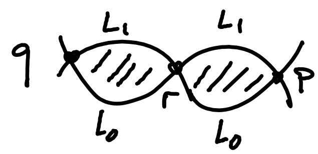
$$\begin{cases} \partial q = 0 \\ \partial p = \left(T^{\omega(u)} - T^{\omega(v)}\right) q \end{cases}$$

If u, v have same area: $\partial = 0$, $HF(L_0, L_1) = \Lambda p \oplus \Lambda q \simeq H^*(S^1; \Lambda)$
(ie. L_0, L_1 Ham. iso.)

If $\omega(u) \neq \omega(v)$ then $HF(L_0, L_1) = 0$
(L_0, L_1 can be disjoined)

Why $\partial^2 = 0$: (still assuming no bubbling):

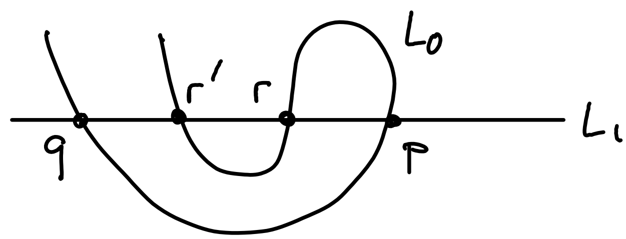
$\mathcal{M}(p, q, [u], J)/\mathbb{R}$ for $\mathrm{ind}(u) = 2$ is expected to be a 1-dim. mfld,
and its boundary consists of broken trajectories



$\partial^2(p)$ counts all broken trajectories $p \to ? \to q$

These come in pairs (ends of index 2 moduli spaces)
& these contributions cancel.

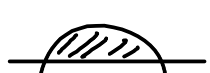
Homework: see this on



Example: $M = T^*N$, $L_0 =$ zero section $\simeq N$
$L_1 = \mathrm{graph}(\varepsilon\, df)$

$\Rightarrow L_0 \cap L_1 = \mathrm{crit}(f)$; $\exists$ grading: $\deg = n - $ Morse index

Assume f Morse-Smale:

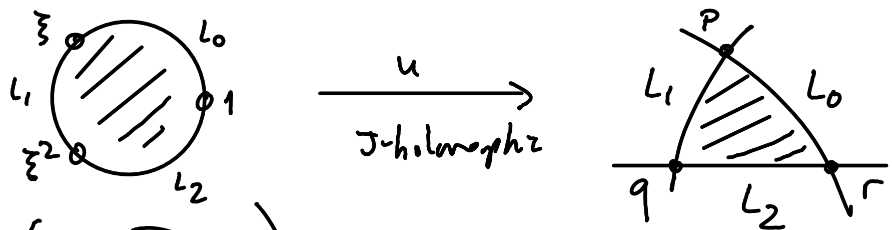
(Floer, Fukaya-Oh) as $\varepsilon \to 0$,  cv to gradient flow lines of f.

Hence $HF^*(L_0,L_1) \simeq HM_{n-*}(f) \otimes \Lambda \simeq H^*(L_0;\Lambda)$.

(Novikov params. play no role since $\int u^*\omega = \varepsilon(f(q)-f(p)) \to 0$)

<u>Product structure:</u> $CF(L_0,L_1) \otimes CF(L_1,L_2) \longrightarrow CF(L_0,L_2)$

Looks at



($\cong$ )

$$p \cdot q = \sum_{\substack{r \in L_0 \cap L_2 \\ \mathrm{ind}(u)=0}} \#\left(\mathcal{M}(p,q,r,[u],J)\right) T^{\omega(u)} r$$